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Solvable Lie algebras with triangular nilradicals

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Abstract. All finite-dimensional indecomposable solvable Lie algebras $L(n, f)$, having the triangular algebra $T(n)$ as their nilradical, are constructed. The number of non-nilpotent elements f in $L(n, f)$ satisfies $1 \leqslant f \leqslant n-1$ and the dimension of the Lie algebra is $\dim L(n, f) = f + \frac{1}{2}n(n-1)$.

Résumé. Toutes les algèbres de Lie résolubles et indécomposables de dimension finie, qui ont un nilradical triangulaire $T(n)$, sont construites. Le nombre d'éléments non nilpotents f dans $L(n, f)$ satisfait $1 \leqslant f \leqslant n-1$ et la dimension de l'algèbre de Lie est $\dim L(n, f) = f + \frac{1}{2}n(n-1)$.

1. Introduction

The purpose of this paper is to construct all indecomposable solvable Lie algebras that have 'triangular algebras' $T(n)$ of dimension $\frac{1}{2}n(n-1)$ $(3 \leqslant n < \infty)$ as their nilradicals. By triangular algebra $T(n)$, we mean the nilpotent Lie algebra isomorphic to the Lie algebra of strictly upper triangular $n \times n$ matrices. The motivation for such a study is manyfold. From a mathematical point of view, this investigation is part of the classification of all finite dimensional Lie algebras. The Levi theorem [1, 2] tells us that every finite-dimensional Lie algebra L is a semidirect sum of a semisimple Lie algebra S and a solvable ideal (the radical R):

$$L = S \triangleright R \qquad [S, S] = S \qquad [S, R] \subseteq R \qquad [R, R] \subset R. \tag{1.1}$$

Semisimple algebras over fields of complex or real numbers have been classified by Cartan [3]. However, the classification of solvable Lie algebras is only complete for low dimensions ($\dim L \leqslant 6$) [4–7]. From Maltsev [8] we know some important results on the structure of Lie algebras, but not on solvable Lie algebras with a given nilradical. More recent articles provided a classification of all Lie algebras with Heisenberg or Abelian nilradicals [9, 10].

As far as physical applications are concerned, we note that solvable Lie algebras often occur as Lie algebras of symmetry groups of differential equations [11]. Group invariant solutions can be obtained by symmetry reduction, using the subalgebras of the symmetry algebra [12]. In this procedure an important step is to identify the symmetry algebra and its subalgebras as abstract Lie algebras. A detailed identification presupposes the existence of a classification of Lie algebras into isomorphism classes.

In section 2, we formulate the problem and the general strategy that we will adopt. In section 3, we illustrate the procedure for the particular case $n = 4$. Guided by this last section, in section 4 we present the general classification for arbitrary n.

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2. Formulation of the problem

2.1. General concepts

Let us first recall some definitions and known results on solvable Lie algebras. A Lie algebra L is solvable, if its derived series DS terminates, i.e.

$$DS = \{L_0 \equiv L, L_1 = [L, L], \ldots, L_k = [L_{k-1}, L_{k-1}] = 0\} \tag{2.1}$$

for some $k \geqslant 0$.

A Lie algebra L is nilpotent, if its central series CS also terminates, i.e.

$$CS = \{L_{(0)} \equiv L, L_{(1)} = [L, L_{(0)}], \ldots, L_{(k)} = [L, L_{(k-1)}] = 0\} \tag{2.2}$$

for some $k \geqslant 0$.

The nilradical $NR(L)$ of a solvable Lie algebra L is the maximal nilpotent ideal of L. This nilradical $NR(L)$ is unique and its dimension satisfies [5]

$$\dim NR(L) \geqslant \tfrac{1}{2} \dim L. \tag{2.3}$$

Any solvable Lie algebra L can be written as the algebraic sum of the nilradical $NR(L)$ and a complementary linear space F

$$L = F \dot{+} NR(L). \tag{2.4}$$

A Lie algebra L is decomposable if it can, by a change of basis, be transformed into a direct sum of two (or more) subalgebras

$$L = L_1 \oplus L_2 \qquad [L_1, L_2] = 0. \tag{2.5}$$

An element N of a Lie algebra L is nilpotent in L if

$$[\ldots[[X, N], N] \ldots N] = 0 \qquad \forall X \in L. \tag{2.6}$$

A set of elements $\{X^\alpha\}$ of L is linearly nilindependent if no nontrivial linear combination of them is nilpotent.

A set of matrices $\{A^\alpha\}_{\alpha=1\ldots n}$ is linearly nilindependent if no nontrivial linear combination of them is a nilpotent matrix, i.e.

$$\left(\sum_{i=1}^{n} c_i A^i\right)^k = 0 \tag{2.7}$$

for some $k \in \mathbb{Z}^+$, implies $c_i = 0 \forall i$.

2.2. Basic structure of the Lie algebra and general strategy

Let us consider the finite triangular algebra $T(n)$ with $n \geqslant 3$ over the field of complex, or real numbers ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$). A basis for this algebra is

$$\begin{aligned} &\{N_{ik} | 1 \leqslant i < k \leqslant n\} \\ &(N_{ik})_{ab} = \delta_{ia}\delta_{kb} \qquad \dim T(n) = \tfrac{1}{2} n(n-1) \equiv r. \end{aligned} \tag{2.8}$$

This basis can be represented by the standard basis of the strictly upper triangular $n \times n$ matrices. The commutation relations are

$$[N_{ik}, N_{ab}] = \delta_{ka} N_{ib} - \delta_{bi} N_{ak}. \tag{2.9}$$

We wish to extend this algebra to an indecomposable solvable Lie algebra $L(n, f)$ of dimension $\frac{1}{2}n(n-1) + f$ having $T(n)$ as its nilradical. In other words, we wish to add f further linearly nilindependent elements to $T(n)$. Let us denote them $\{X^1, \ldots, X^f\}$.

The derived algebra $[L, L]$ of a solvable Lie algebra L is contained in its nilradical [2]. The commutation relations will have the form

$$[X^\alpha, N_{ik}] = A^\alpha_{ik,pq} N_{pq} \tag{2.10}$$
$$[X^\alpha, X^\beta] = \sigma^{\alpha\beta}_{pq} N_{pq} \tag{2.11}$$
$$1 \leqslant \alpha, \beta \leqslant f \leqslant r \qquad A^\alpha_{ik,pq},\ \sigma^{\alpha\beta}_{pq} \in \mathbb{K}.$$

(Here and in the rest of the paper, we use the Einstein summation convention over repeated indices, unless stated otherwise). The commutation relations (2.10) can be rewritten as

$$\begin{aligned} &[X^\alpha, N] = A^\alpha N \\ &N \equiv (N_{12}N_{23}\ldots N_{(n-1)n}N_{13}\ldots N_{(n-2)n}\ldots N_{1n})^{\mathrm{T}} \\ &A^\alpha \in \mathbb{K}^{r\times r} \qquad N \in \mathbb{K}^{r\times 1}. \end{aligned} \tag{2.12}$$

(The superscript T indicates transposition.) Here, N is a 'column vector' of basis elements of $NR(L)$ ordered by first taking the elements $N_{i(i+1)}$ next to the diagonal, then $N_{i(i+2)}$ (removed two steps from the diagonal) etc. We shall call the matrices A^α 'structure matrices'.

A classification of the Lie algebras $L(n, f)$ thus amounts to a classification of the structure matrices A^α and the constants $\sigma^{\alpha\beta}_{pq}$. The Jacobi identities have to be respected by the following three types of triplets (those with three N's are satisfied automatically)

$$\begin{aligned} &\{X^\alpha, N_{ik}, N_{ab}\} f \geqslant 1 \qquad \{X^\alpha, X^\beta, N_{ik}\} f \geqslant 2 \qquad \{X^\alpha, X^\beta, X^\gamma\} f \geqslant 3 \\ &1 \leqslant i < k \leqslant n \qquad 1 \leqslant a < b \leqslant n \qquad 1 \leqslant \alpha, \beta, \gamma \leqslant f. \end{aligned} \tag{2.13}$$

Which give us respectively the three equations

$$\delta_{ka} A^\alpha_{ib,pq} N_{pq} - \delta_{bi} A^\alpha_{ak,pq} N_{pq} + A^\alpha_{ik,bq} N_{aq} - A^\alpha_{ik,pa} N_{pb} + A^\alpha_{ab,pi} N_{pk} - A^\alpha_{ab,kq} N_{iq} = 0 \tag{2.14}$$
$$[A^\alpha, A^\beta]_{ik,pq} N_{pq} = \sigma^{\alpha\beta}_{kq} N_{iq} - \sigma^{\alpha\beta}_{pi} N_{pk} \tag{2.15}$$
$$\sigma^{\alpha\beta}_{pq} A^\gamma_{pq,ik} + \sigma^{\gamma\alpha}_{pq} A^\beta_{pq,ik} + \sigma^{\beta\gamma}_{pq} A^\alpha_{pq,ik} = 0. \tag{2.16}$$

The equation (2.14) will give restrictions on the form of the structure matrices A^α. We will transform these matrices into a 'canonical' form by transformations that leave the commutation relations (2.9) of the $NR(L)$ invariant, but transform the matrices A^α and the constants $\sigma^{\alpha\beta}_{pq}$. These transformations will be

(i) redefinition of all the non-nilpotent elements:

$$\begin{aligned} &X^\alpha \longrightarrow X^\alpha + \mu^\alpha_{pq} N_{pq} \qquad \mu^\alpha_{pq} \in \mathbb{K} \\ &\Rightarrow A^\alpha_{ik,ab} \longrightarrow A^\alpha_{ik,ab} + \delta_{kb}\mu^\alpha_{ai} - \delta_{ia}\mu^\alpha_{kb} \end{aligned} \tag{2.17}$$

(ii) change of basis in $NR(L)$:

$$\begin{aligned} &N \longrightarrow GN \qquad G \in GL(r, \mathbb{K}) \\ &\Rightarrow A^\alpha \longrightarrow G A^\alpha G^{-1} \end{aligned} \tag{2.18}$$

(iii) linear combinations of the elements X^α and hence of the matrices A^α.

Note that the element N_{1n} does not contribute in the transformation (2.17) since it commutes with all the elements in the $NR(L)$. Thus μ^α_{1n} is not used in (2.17). Also, the matrix G has to be suitably restricted in order to preserve the commutation relations (2.9) of the $NR(L)$.

From the equations (2.15) and (2.16) we obtain some relations between the matrices A^α and the constants $\sigma_{pq}^{\alpha\beta}$. By exploiting the fact that μ_{1n}^α is not utilized in (2.17), for $f \geqslant 2$ we will make an additional transformation to simplify the constants $\sigma_{pq}^{\alpha\beta}$, i.e.

$$\begin{aligned} & X^\alpha \longrightarrow X^\alpha + \mu_{1n}^\alpha N_{1n} \qquad \mu_{1n}^\alpha \in \mathbb{K} \\ & \Rightarrow \sigma_{pq}^{\alpha\beta} \longrightarrow \sigma_{pq}^{\alpha\beta} + \mu_{1n}^\beta A_{1n,pq}^\alpha - \mu_{1n}^\alpha A_{1n,pq}^\beta . \end{aligned} \tag{2.19}$$

(Where $A^\alpha \longrightarrow A^\alpha$, i.e. the structure matrices stay the same.) It will therefore be possible to simplify some constants $\sigma_{pq}^{\alpha\beta}$ associated with the matrices A^α, A^β.

3. Illustration of the procedure for low dimensions

3.1. The case $n = 3$

In this case, the Lie algebra $T(3)$ is isomorphic to the Heisenberg algebra $H(1)$. As mentioned previously, solvable Lie algebras with Heisenberg nilradicals were classified earlier [9]. We will therefore consider $n > 3$ from this point on. The dimension $n = 3$ is the only case for which there is an isomorphism between the triangular and the Heisenberg Lie algebras.

3.2. The case $n = 4$

In this particular case, we have

$$A^\alpha \in \mathbb{K}^{6\times 6} \qquad N = (N_{12} N_{23} N_{34} N_{13} N_{24} N_{14})^{\mathrm{T}}. \tag{3.1}$$

Let us first consider relations (2.14). We can separate them into two classes of equations. The first arises from the triplets $\{X^\alpha, N_{ik}, N_{kb}\}$, $1 \leqslant i < k = a < b \leqslant 4$, which give

$$A_{ib,pq}^\alpha N_{pq} + A_{ik,bq}^\alpha N_{kq} - A_{ik,pk}^\alpha N_{pb} + A_{kb,pi}^\alpha N_{pk} - A_{kb,kq}^\alpha N_{iq} = 0 \quad \text{(no summation over k)}. \tag{3.2}$$

The second class comes from the triplets $\{X^\alpha, N_{ik}, N_{ab}\}$, $1 \leqslant i < k \leqslant 4$, $1 \leqslant a < b \leqslant 4$, $k \neq a (b \neq i)$ and in this case equation (2.14) becomes

$$A_{ik,bq}^\alpha N_{aq} - A_{ik,pa}^\alpha N_{pb} + A_{ab,pi}^\alpha N_{pk} - A_{ab,kq}^\alpha N_{iq} = 0. \tag{3.3}$$

We begin by considering equation (3.3). From each possible triplet associated with this class of equation, we use the linear independence of the $\{N_{lm}\}$ to determine relations between the elements of A^α. For example, from the triplet $\{X^\alpha, N_{12}, N_{34}\}$, we obtain

$$A_{12,13}^\alpha + A_{34,24}^\alpha = 0 \qquad A_{12,23}^\alpha = A_{34,23}^\alpha = 0. \tag{3.4}$$

When we apply equation (3.3) to the 11 triplets associated to this equation, we find

$$A^\alpha = \begin{pmatrix} * & 0 & A_{12,34}^\alpha & A_{12,13}^\alpha & * & * \\ 0 & * & 0 & * & * & * \\ A_{34,12}^\alpha & 0 & * & * & -(A_{12,13}^\alpha) & * \\ 0 & 0 & 0 & * & (A_{12,34}^\alpha) & * \\ 0 & 0 & 0 & (A_{34,12}^\alpha) & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}. \tag{3.5}$$

Where $*$ denote arbitrary elements unrelated to others in the matrices A^α.

In the same manner, we apply equation (3.2) to the four triplets associated with this class of equation. This gives us some further relations between the matrix elements and A^α becomes

$$A^\alpha = \begin{pmatrix} A^\alpha_{12,12} & 0 & 0 & A^\alpha_{12,13} & * & * \\ & A^\alpha_{23,23} & 0 & A^\alpha_{23,13} & A^\alpha_{23,24} & * \\ & & A^\alpha_{34,34} & * & -(A^\alpha_{12,13}) & * \\ & & & A^\alpha_{12,12} + A^\alpha_{23,23} & 0 & (A^\alpha_{23,24}) \\ & & & & A^\alpha_{23,23} + A^\alpha_{34,34} & (A^\alpha_{23,13}) \\ & & & & & A^\alpha_{12,12} + A^\alpha_{23,23} + A^\alpha_{34,34} \end{pmatrix}. \quad (3.6)$$

To simplify the form of the matrix (3.6), we carry out the transformation (2.17) for the f matrices A^α. Given the liberty of the five constants μ^α_{pq} for each α independently (the sixth one, μ^α_{14} does not contribute), we can arrange to have

$$A^\alpha_{12,13} = A^\alpha_{12,14} = A^\alpha_{23,13} = A^\alpha_{23,24} = A^\alpha_{34,14} = 0. \quad (3.7)$$

Therefore, each matrix A^α can be transformed into

$$A^\alpha = \begin{pmatrix} A^\alpha_{12,12} & 0 & 0 & 0 & A^\alpha_{12,24} & 0 \\ & A^\alpha_{23,23} & 0 & 0 & 0 & A^\alpha_{23,14} \\ & & A^\alpha_{34,34} & A^\alpha_{34,13} & 0 & 0 \\ & & & A^\alpha_{13,13} & 0 & 0 \\ & & & & A^\alpha_{24,24} & 0 \\ & & & & & A^\alpha_{14,14} \end{pmatrix}$$
$$A^\alpha_{ik,ik} = \sum_{p=i}^{k-1} A^\alpha_{p(p+1),p(p+1)}. \quad (3.8)$$

These matrices must be linearly nilindependent otherwise the $NR(L)$ would be larger than $T(4)$. In particular, this implies that we cannot simultaneously have $A^\alpha_{12,12} = A^\alpha_{23,23} = A^\alpha_{34,34} = 0$. Also, since we have three parameters on the diagonal, the nilindependence between the A^α implies that we have at most three non-nilpotent elements, i.e.

$$1 \leqslant f \leqslant 3. \quad (3.9)$$

Let us now look at the cases $f \geqslant 2$. The structure matrices A^α have the 'canonical' form given by (3.8), therefore the possibly nonzero elements of the commutators $[A^\alpha, A^\beta]$ are

$$[A^\alpha, A^\beta]_{12,24} \qquad [A^\alpha, A^\beta]_{23,14} \qquad [A^\alpha, A^\beta]_{34,13}. \quad (3.10)$$

By the linear independence of the $\{N_{lm}\}$ and from (2.15), (3.10) and (2.11) we find that

$$[A^\alpha, A^\beta] = 0 \quad (3.11)$$
$$[X^\alpha, X^\beta] = \sigma^{\alpha\beta} N_{14}. \quad (3.12)$$

Finally, we consider the case $f = 3$. In view of the 'canonical' form of the structure matrices A^α and by relation (3.12), equation (2.16) becomes

$$\sigma^{12} A^3_{14,14} + \sigma^{31} A^2_{14,14} + \sigma^{23} A^1_{14,14} = 0. \quad (3.13)$$

Moreover, the transformation (2.19) will modify the constants $\sigma^{\alpha\beta}$ into

$$\sigma^{\alpha\beta} \longrightarrow \sigma^{\alpha\beta} + \mu^\beta_{14} A^\alpha_{14,14} - \mu^\alpha_{14} A^\beta_{14,14}. \quad (3.14)$$

Hence, by using (3.14) for $f = 2$ and (3.14), (3.13) for $f = 3$, we obtain

$$[X^\alpha, X^\beta] = \begin{cases} \sigma^{\alpha\beta} N_{14} & \text{for } A^1_{14,14} = \cdots = A^f_{14,14} = 0 \\ 0 & \text{otherwise.} \end{cases} \quad (3.15)$$

To further simplify the structure matrix, let us perform a transformation of the type (2.18), i.e.

$$N \longrightarrow G_1 N \qquad G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & g_1 & 0 \\ & 1 & 0 & 0 & 0 & g_2 \\ & & 1 & g_3 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}. \tag{3.16}$$

This transformation leaves the commutation relations (2.9) of the $NR(L)$ invariant, but transforms the matrices $A^\alpha \longrightarrow G_1 A^\alpha G_1^{-1}\ \forall\alpha$, i.e.

$$\begin{aligned} A^\alpha_{12,24} &\longrightarrow A^\alpha_{12,24} + g_1(A^\alpha_{23,23} + A^\alpha_{34,34} - A^\alpha_{12,12}) \\ A^\alpha_{23,14} &\longrightarrow A^\alpha_{23,14} + g_2(A^\alpha_{12,12} + A^\alpha_{34,34}) \\ A^\alpha_{34,13} &\longrightarrow A^\alpha_{34,13} + g_3(A^\alpha_{12,12} + A^\alpha_{23,23} - A^\alpha_{34,34}). \end{aligned} \tag{3.17}$$

Thus using g_1 we can eliminate $A^\nu_{12,24}(1 \leqslant \nu \leqslant f)$ of the specific matrix A^ν, provided that $A^\nu_{23,23} + A^\nu_{34,34} \neq A^\nu_{12,12}$. The constants g_2 and g_3 are used in the same way. Therefore, at most, three off-diagonal elements can be eliminated by this transformation.

Now, we carry out a second transformation G_2 of the $NR(L)$, such that the total transformation will be given by $G = G_2 G_1$. The matrix G_2 is diagonal and the commutation relations (2.9) of the $NR(L)$ are left invariant for a transformation of the type

$$N \longrightarrow G_2 N, G_2 = \begin{pmatrix} g_{12} & & & & & \\ & g_{23} & & & & \\ & & g_{34} & & & \\ & & & g_{12}g_{23} & & \\ & & & & g_{23}g_{34} & \\ & & & & & g_{12}g_{23}g_{34} \end{pmatrix} g_{ik} \in \mathbb{K} \setminus \{0\}. \tag{3.18}$$

The matrices A^α are transformed as $A^\alpha \longrightarrow G_2 A^\alpha G_2^{-1}\ \forall\alpha$, where

$$A^\alpha_{12,24} \longrightarrow \left(\frac{g_{12}}{g_{24}}\right) A^\alpha_{12,24} \qquad A^\alpha_{23,14} \longrightarrow \left(\frac{g_{23}}{g_{14}}\right) A^\alpha_{23,14} \qquad A^\alpha_{34,13} \longrightarrow \left(\frac{g_{34}}{g_{13}}\right) A^\alpha_{34,13}. \tag{3.19}$$

Therefore, we can normalize up to three nonzero off-diagonal elements. For $\mathbb{K} = \mathbb{C}$ they can be set equal to $+1$, for $\mathbb{K} = \mathbb{R}$ we must in some cases allow the possibility of normalizing to either $+1$, or to -1.

3.2.1. The Lie algebras $L(4,1)$. The matrix A has the 'canonical' form given by (3.8). Using the transformation (3.17), we can eliminate all off-diagonal elements, unless the diagonal elements satisfy specific equations (e.g. $A_{23,23} + A_{34,34} - A_{12,12} = 0$). At most, two of their equalities can hold, otherwise the matrix A will be nilpotent. Thus, at most two off-diagonal entries remain. They can be normalized to $+1$, with one exception, namely, if we have $A_{12,24} \neq 0$, $A_{34,13} \neq 0$ for $\mathbb{K} = \mathbb{R}$. Then, we can transform to one of the following:

$$A_{12,24} = +1 \qquad A_{34,13} = \pm 1$$

($A_{34,13} = -1$ is not equivalent to $A_{34,13} = +1$).

The final result is that for $\mathbb{K} = \mathbb{C}$, 12 inequivalent types of such matrices exist, one of them depending on two complex parameters, four depending on one complex parameter, seven without parameters. For $\mathbb{K} = \mathbb{R}$, altogether 13 types exist. Among them, one depends on two real parameters, four on one real parameter and eight without parameters. These matrices are listed in table A1 in the appendix. The set of inequivalent matrices A in fact represent all the possible Lie algebras $L(4,1)$ of dimension seven. For $\mathbb{K}=\mathbb{C}$ the algebra $R_{1,13}$ is equivalent to $K_{1,12}$.

3.2.2. The Lie algebras $L(4,2)$. From the equations (3.11) and (3.15) and from the previous results on the matrices A^α, we can now determine the different types of Lie algebras $L(4,2)$ (of dimension eight). For $\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$, 10 inequivalent types of such algebras exist, one depending on two parameters, five on one parameter, four without parameters. These Lie algebras are presented in table A2 in the appendix.

3.2.3. The Lie algebra $L(4,3)$. We can choose a basis for the set of matrices $\{A^1, A^2, A^3\}$, by putting $A^\alpha_{j(j+1),j(j+1)} = \delta_{\alpha j}(\alpha, j = 1, 2, 3)$ in the general 'canonical' form. For the matrix A^1, we use the transformation (3.17) to annul $A^1_{12,24}$, $A^1_{23,14}$ and $A^1_{34,13}$. The commutativity of the matrices A^α imposes that the matrices A^2 and A^3 are also diagonal.

Since $A^1_{14,14}, A^2_{14,14}, A^3_{14,14}$ are different from zero, we can use equation (3.14) to put $[X^1, X^2] = [X^2, X^3] = 0$ and $[X^3, X^1] = \sigma^{31}N_{14}$. The relation (3.13) then imposes $\sigma^{31} = 0$ and the commutation relations for the non-nilpotent elements become

$$[X^\alpha, X^\beta] = 0 \qquad \alpha, \beta = 1, 2, 3. \tag{3.20}$$

Therefore, there exists only one Lie algebra $L(4,3)$ ($\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$) and its dimension is nine. This algebra is given in table A3 in the appendix.

The results of section 3.2 can be summed up as a theorem.

Theorem 1. Every solvable Lie algebra $L(4,f)$ with a six-dimensional triangular nilradical $T(4)$ can be transformed to a canonical basis $\{X^\alpha, N_{ik}\}$, $\alpha = 1,\ldots,f$, $1 \leqslant i < k \leqslant 4$, $1 \leqslant f \leqslant 3$. The commutation relations in this basis are given by equation (2.9),(2.10) and (2.11). The structure matrices A^α all have the form (3.8).

For $f = 1$ the matrix $A^1 \equiv A$ has one of the forms given in table A1.

For $f = 2$ the matrices $\{A^1, A^2\}$ have one of the forms given in table A2. The elements $\{X^1, X^2\}$ commute in all cases except K_2 of table A2, when σ is a nonzero arbitrary constant.

For $f = 3$ there is precisely one such algebra, given by the matrices $\{A^1, A^2, A^3\}$ of table A3, with all elements $\{X^1, X^2, X^3\}$ commuting.

Every algebra $L(4,f)$ is isomorphic to precisely one algebra in Table Af, for $f = 1, 2, 3$, respectively.

4. Solvable Lie algebras $L(n,f)$ for $n \geqslant 4$

4.1. General results

Lemma 1. The structure matrices $A^\alpha = \{A^\alpha_{ik,ab}\}$, $1 \leqslant i < k \leqslant n$, $1 \leqslant a < b \leqslant n$ have the following properties.

(1) They are upper triangular.

(2) The only off-diagonal matrix elements that do not vanish identically and cannot be annuled by a redefinition of the elements X^α are:

$$A^\alpha_{12,2n} \qquad A^\alpha_{j(j+1),1n}(2 \leqslant j \leqslant n-2) \qquad A^\alpha_{(n-1)n,1(n-1)}. \tag{4.1}$$

(3) The diagonal elements $A^{\alpha}_{i(i+1),i(i+1)}$, $1 \leqslant i \leqslant n-1$ are free. The other diagonal elements satisfy

$$A^{\alpha}_{ik,ik} = \sum_{p=i}^{k-1} A^{\alpha}_{p(p+1),p(p+1)} \qquad k > i+1. \tag{4.2}$$

Proof. We shall use relations (2.14) that are consequences of the Jacobi relations for $\{X^{\alpha}, N_{ik}, N_{ab}\}$. Let us prove each statement in the theorem separately.

(1) All matrix elements below the diagonal vanish identically, i.e.

$$A^{\alpha}_{ik,ab} = 0 \qquad \text{for} \begin{cases} k-i > b-a \\ k-i = b-a \qquad i > a. \end{cases} \tag{4.3}$$

We prove this statement by induction. In section 3, we have shown that lemma 1 is valid for $n = 4$. Now let us assume it is valid for $n = N-1 \geqslant 4$ and prove that it is then also valid for $n = N$. By the induction assumption, we have

$$A^{\alpha}_{lm,pq} = 0 \qquad \text{for} \begin{cases} m-l > q-p \\ m-l = q-p \qquad l > p. \end{cases} \tag{4.4}$$
$$1 \leqslant l < m \leqslant N-1 \qquad 1 \leqslant p < q \leqslant N-1.$$

Now consider $n = N$. We are adding new entries in old rows $A^{\alpha}_{ik,aN}$, new entries in old columns $A^{\alpha}_{iN,ab}$ and new rows intersecting new columns $A^{\alpha}_{iN,aN}$ (here, lower case labels vary from 1 to $N-1$). We must show that all news entries also vanish.

Let us first take (2.14) for $k = a = i+1$, $1 \leqslant i \leqslant N-2$, $i+2 \leqslant b \leqslant N$. The coefficient of N_{pq} for $p \geqslant i+2$ provides the identities

$$A^{\alpha}_{ib,pq} = 0. \tag{4.5}$$

In particular we obtain

$$A^{\alpha}_{ib,pN} = 0 \qquad b-i > N-p \tag{4.6}$$

which means that we have no nonzero entries in new columns and old rows. Indeed, the smallest possible value of $Z \equiv k-i+a-N$ for which $A^{\alpha}_{ib,pN}$ is not forced to be zero by equation (4.5) is reached for $b = i+1$, $p = N-1$ or for $p = i+1$, $b = N-2$. In both cases, the element $A^{\alpha}_{ib,pN}$ is above the diagonal.

Now consider equation (2.14) for $k = a = N-1, b = N, 1 \leqslant i \leqslant N-2$. The coefficients of N_{pq} for $q \leqslant N-2$, $N_{p(N-1)}$ and N_{pN} yield, in particular

$$A^{\alpha}_{iN,pq} = 0 \qquad q \leqslant N-2 \tag{4.7}$$
$$A^{\alpha}_{iN,p(N-1)} = 0 \qquad p \geqslant i \tag{4.8}$$
$$A^{\alpha}_{iN,p(N-1)} + A^{\alpha}_{(N-1)N,pi} = 0 \tag{4.9}$$
$$A^{\alpha}_{i(N-1),p(N-1)} - A^{\alpha}_{iN,pN} = 0 \qquad p \neq i. \tag{4.10}$$

Note that equation (4.8) is obtained from equation (4.9). We have $A^{\alpha}_{i(N-1),p(N-1)} = 0$ for $p > i$ by the induction hypothesis. Hence, also $A^{\alpha}_{iN,pN} = 0$ by equation (4.10). The remaining elements in new rows below the diagonal are $A^{\alpha}_{iN,(i-1)(N-1)}$ and $A^{\alpha}_{(N-1)N,(i-1)i}$ with $2 \leqslant i \leqslant N-1$. Moreover, these elements are related by equation (4.9) for $2 \leqslant i \leqslant N-2$. Let us now use relation (2.14) for $k = i+1, a = N-1, b = N, 1 \leqslant i \leqslant N-3$. The coefficient of N_{iq} for $q \leqslant N-1$ must vanish, hence $A^{\alpha}_{(N-1)N,(i+1)q} = 0$ which can be rewritten as

$$A^{\alpha}_{(N-1)N,(i-1)i} \qquad 3 \leqslant i \leqslant N-1. \tag{4.11}$$

The coefficient of N_{13} for $i = 2$ must vanish, hence

$$A^{\alpha}_{(N-1)N,12} = 0. \tag{4.12}$$

Relation (4.9) then implies $A^{\alpha}_{iN,(i-1)(N-1)} = 0 (2 \leqslant i \leqslant N-2)$ and this completes the proof of the statement that A^{α} is upper triangular.

(2) Let us now consider the matrix elements $A^{\alpha}_{ik,ab}$ above the diagonal. First of all, we note the relations:

$$A^{\alpha}_{ib,pb} - A^{\alpha}_{ik,pk} = 0 \qquad p \neq i,\ 1 \leqslant i < k < b \leqslant n \tag{4.13}$$

$$A^{\alpha}_{ib,iq} - A^{\alpha}_{kb,kq} = 0 \qquad q \neq b,\ 1 \leqslant i < k < b \leqslant n \tag{4.14}$$

$$A^{\alpha}_{ik,ia} + A^{\alpha}_{ab,kb} = 0 \qquad k \neq a,\ b \neq i,\ 1 \leqslant i < k \leqslant n,\ 1 \leqslant a < b \leqslant n. \tag{4.15}$$

Relations (4.13) and (4.14) follow from (2.14) with $k = a$, (4.15) from (2.14) with $k \neq a, i \neq b$.

Now let us use the transformation (2.17) to annul certain off-diagonal elements. Specifically, we use the coefficient μ^{α}_{pq} in the following manner:

$$\begin{aligned} &\mu^{\alpha}_{1m} : A^{\alpha}_{m(m+1),1(m+1)} \longrightarrow 0 \qquad 2 \leqslant m \leqslant n-1 \\ &\mu^{\alpha}_{lm} : A^{\alpha}_{(l-1)l,(l-1)m} \longrightarrow 0 \qquad 2 \leqslant l \leqslant n-1,\ l+1 \leqslant m \leqslant n. \end{aligned} \tag{4.16}$$

Note that μ^{α}_{1n} was not used and remains free for future use. Furthermore, combining (4.16) with (4.13)–(4.15) we obtain many more zeros in the matrix A^{α}.

Using relations (2.14) for $k = i+1 \neq a, b \neq i, 1 \leqslant i \leqslant n-1, 1 \leqslant a < b \leqslant n$ we find the relations

$$\begin{aligned} &A^{\alpha}_{i(i+1),bq} - A^{\alpha}_{ib,(i+1)q} = 0 \\ &A^{\alpha}_{ab,(i+1)q} = 0 \qquad i \neq a \neq i+1,\ q \neq b \neq i \\ &A^{\alpha}_{i(i+1),pa} = 0 \qquad a \neq i+1,\ p \neq i \\ &A^{\alpha}_{i(i+1),bq} = 0 \qquad q \neq i+1,\ b \neq i. \end{aligned} \tag{4.17}$$

We still need information on the elements $A^{\alpha}_{in,ab}, A^{\alpha}_{ik,an}$. For this we consider equation (2.14) for $k = a = i+1 < b \leqslant n (1 \leqslant i \leqslant n-2, i+2 \leqslant b \leqslant n)$. We obtain

$$\begin{aligned} &A^{\alpha}_{ib,pq} = 0 \qquad i \neq p \neq i+1,\ i+1 \neq q \neq b \\ &A^{\alpha}_{ib,p(i+1)} + A^{\alpha}_{(i+1)b,pi} = 0. \end{aligned} \tag{4.18}$$

Together, relations (4.13)–(4.18) give us zeros everywhere exept for the elements (4.1). These elements never enter into equation (2.14) exept for some identically respected trivial triplets of the types $\{X^{\alpha}, N_{ik}, N_{ik}\}$. Therefore, they are all free and this completes the proof of the second affirmation in lemma 1.

(3) To obtain relations between the diagonal elements, take $1 \leqslant i < k = a < b \leqslant n$ in equation (2.14). The coefficient of N_{ib} is

$$A^{\alpha}_{ib,ib} - A^{\alpha}_{ik,ik} - A^{\alpha}_{kb,kb} = 0. \tag{4.19}$$

Choosing $a = i+1$, $b = i+2$ $(1 \leqslant i \leqslant n-2)$, we obtain

$$A^{\alpha}_{i(i+2),i(i+2)} = A^{\alpha}_{i(i+1),i(i+1)} + A^{\alpha}_{(i+1)(i+2),(i+1)(i+2)} = 0. \tag{4.20}$$

Now choosing $a = i+2, b = i+3 (1 \leqslant i \leqslant n-3)$, we obtain

$$A^{\alpha}_{i(i+3),i(i+3)} = A^{\alpha}_{i(i+1),i(i+1)} + A^{\alpha}_{(i+1)(i+2),(i+1)(i+2)} + A^{\alpha}_{(i+2)(i+3),(i+2)(i+3)} = 0. \tag{4.21}$$

Proceeding recursively, we deduce relation (4.2) and this completes the proof of statement 3 of lemma 1. □

Lemma 2. The maximal number of non-nilpotent elements is

$$f_{\max} = n - 1. \tag{4.22}$$

Proof. The proof is straightfoward since we have a maximum of $n-1$ parameters on the diagonal and we impose the nilindependence between the matrices A^α. □

Up to now we have considered the case $f \geqslant 1$ and this gave us lemma 1 describing each of the matrices A^α. Now, we shall consider the cases $f \geqslant 2$ and $f \geqslant 3$ and must also satisfy the equations (2.15) and (2.16).

Let us first consider $f \geqslant 2$. From lemma 1, the possibly nonzero elements of the commutators $[A^\alpha, A^\beta]$ are

$$[A^\alpha, A^\beta]_{12,2n} \qquad [A^\alpha, A^\beta]_{j(j+1),1n}\ (j = 2, \ldots, n-2) \qquad [A^\alpha, A^\beta]_{(n-1)n,1(n-1)}. \tag{4.23}$$

Therefore, from (2.15), (4.23) and (2.11) we find that

$$[A^\alpha, A^\beta] = 0 \tag{4.24}$$

$$[X^\alpha, X^\beta] = \sigma^{\alpha\beta} N_{1n}. \tag{4.25}$$

Finally, we consider $f \geqslant 3$. From equation (4.25) and lemma 1, equation (2.16) reduces to

$$\sigma^{\alpha\beta} A^\gamma_{1n,1n} + \sigma^{\gamma\alpha} A^\beta_{1n,1n} + \sigma^{\beta\gamma} A^\alpha_{1n,1n} = 0. \tag{4.26}$$

Lemma 3. The commutation relations between the structure matrices and the non-nilpotent elements can be transformed to a canonical form satisfying

$$[A^\alpha, A^\beta] = 0 \tag{4.27}$$

$$[X^\alpha, X^\beta] = \begin{cases} \sigma^{\alpha\beta} N_{1n} & \text{for } A^1_{1n,1n} = \cdots = A^f_{1n,1n} = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.28}$$

Proof. The commutation relations between the structure matrices have been proven already, so we only consider the proof of equation (4.28). Using lemma 1 and transformation (2.19), we modify the constants $\sigma^{\alpha\beta}$ to

$$\sigma^{\alpha\beta} \longrightarrow \sigma^{\alpha\beta} + \mu^\beta_{1n} A^\alpha_{1n,1n} - \mu^\alpha_{1n} A^\beta_{1n,1n}. \tag{4.29}$$

Unless we have $A^1_{1n,1n} = \cdots = A^f_{1n,1n} = 0$, this transformation can be used to cancel $(f-1)$ constants $\sigma^{\alpha\beta}$. The remaining constants are forced to be zeros by equation (4.26) and this completes the proof. □

4.2. Changes of basis in the nilradical

As in the case $n = 4$, we want to further simplify the structure matrices. For this, we generalize to $n > 4$ the previous transformations G_1 and G_2 which transform the $NR(L)$, but preserve its commutation relations. The transformation G_1 is given by

$$N \longrightarrow G_1 N \qquad (G_1)_{ab,pq} = \delta_{ab,pq} + \underbrace{\Delta_{ab,pq} g_a}_{\text{no sum. over } a} \qquad g_a \in \mathbb{K} \tag{4.30}$$

where

$$\delta_{ab,pq} \equiv \delta_{ap}\delta_{bq}$$

$$\Delta_{ab,pq} \equiv \delta_{ab,12}\delta_{pq,2n} + \left(\sum_{j=2}^{n-2}\delta_{ab,j(j+1)}\right)\delta_{pq,1n} + \delta_{ab,(n-1)n}\delta_{pq,1(n-1)}. \tag{4.31}$$

Note that with this definition the elements of A^α satisfy $A^\alpha_{ab,pq} = (\delta_{ab,pq} + \Delta_{ab,pq})A^\alpha_{ab,pq}$. The transformation preserves the commutation relations in the $NR(L)$ and the matrices A^α are transformed as $A^\alpha \longrightarrow G_1 A^\alpha G_1^{-1} \forall\alpha$. The diagonal elements are invariant and the off-diagonal ones transform as

$$\begin{aligned}
&A^\alpha_{12,2n} \longrightarrow A^\alpha_{12,2n} + g_1(A^\alpha_{2n,2n} - A^\alpha_{12,12})\\
&A^\alpha_{j(j+1),1n} \longrightarrow A^\alpha_{j(j+1),1n} + g_j(A^\alpha_{1n,1n} - A^\alpha_{j(j+1),j(j+1)}) \qquad j = 2,\dots,n-2\\
&A^\alpha_{(n-1)n,1(n-1)} \longrightarrow A^\alpha_{(n-1)n,1(n-1)} + g_{n-1}(A^\alpha_{1(n-1),1(n-1)} - A^\alpha_{(n-1)n,(n-1)n})
\end{aligned} \tag{4.32}$$

with

$$A^\alpha_{ik,ik} = \sum_{p=i}^{k-1} A^\alpha_{p(p+1),p(p+1)}.$$

As in the case $n = 4$, the constants $g_m (m = 1,\dots,n-1)$ can be used to eliminate up to $(n-1)$ off-diagonal elements of the matrices A^α.

Lemma 4. The canonical form of a structure matrix A^α has a nonzero off-diagonal element $A^\alpha_{ik,ab}$ only if

$$A^\beta_{ab,ab} = A^\beta_{ik,ik} \qquad \beta = 1,\dots,f. \tag{4.33}$$

This is true simultaneously for all β.

Proof. The off-diagonal element $A^\alpha_{ik,ab}$ of a given matrix A^α can be transformed to zero by transformation (4.32), unless we have $A^\alpha_{ab,ab} = A^\alpha_{ik,ik}$. Now let us consider a second matrix A^β. The relation $[A^\alpha, A^\beta] = 0$ among the structure matrices implies

$$A^\alpha_{ik,ab}(A^\beta_{ab,ab} - A^\beta_{ik,ik}) = A^\beta_{ik,ab}(A^\alpha_{ab,ab} - A^\alpha_{ik,ik}), \tag{4.34}$$

so if $A^\alpha_{ik,ab} \neq 0$, we must have $A^\beta_{ab,ab} = A^\beta_{ik,ik}$ $\forall\beta$. □

Now, consider the second transformation G_2 given by

$$N \longrightarrow G_2 N \qquad (G_2)_{ab,pq} = \delta_{ab,pq} g_{ab} \qquad g_{ab} \in \mathbb{K}\setminus\{0\}. \tag{4.35}$$

This transformation preserves the commutation relations in $NR(L)$ if

$$g_{ab} = \prod_{p=a}^{b-1} g_{p(p+1)}. \tag{4.36}$$

The matrices A^α are transformed as $A^\alpha \longrightarrow G_2 A^\alpha G_2^{-1} \forall\alpha$. The off-diagonal elements are transformed as

$$\begin{aligned}
&A^\alpha_{12,2n} \longrightarrow \left(\frac{g_{12}}{g_{2n}}\right) A^\alpha_{12,2n}\\
&A^\alpha_{j(j+1),1n} \longrightarrow \left(\frac{g_{j(j+1)}}{g_{1n}}\right) A^\alpha_{j(j+1),1n} \qquad j = 2,\dots,n-2\\
&A^\alpha_{(n-1)n,1(n-1)} \longrightarrow \left(\frac{g_{(n-1)n}}{g_{1(n-1)}}\right) A^\alpha_{(n-1)n,1(n-1)}.
\end{aligned} \tag{4.37}$$

This transformation is used to normalize the nonzero off-diagonal elements to $+1$ for $\mathbb{K} = \mathbb{C}$ and to $+1$, or possibly -1 for $\mathbb{K} = \mathbb{R}$. Up to $n-1$ elements can be normalized since we have $n-1$ independent entries in G_2 (see equation (4.36)).

Note that before the normalization of the off-diagonal elements, we can normalize to $+1$ the first nonzero entry on diagonal of each matrix A^α (we choose the first nonzero one).

4.3. The Lie algebras $L(n, 1)$

Let us consider one extreme case, namely $f = 1$, for which we obtain the following lemma.

Lemma 5. The structure matrix $A = \{A_{ik,ab}\} 1 \leqslant i < k \leqslant n, 1 \leqslant a < b \leqslant n$ of the Lie algebra $L(n, 1)$ has the following properties.

(1) The maximum number of off-diagonal elements is $n-2$.

(2) The off-diagonal elements can all be normalized to $+1$ for $\mathbb{K} = \mathbb{C}$ and to $+1$, or -1 for $\mathbb{K} = \mathbb{R}$.

Proof. Let us prove each statement in the theorem separately.

(1) First, suppose that we have $n-1$ nonzero off-diagonal elements. The off-diagonal elements remain different from zero if all the terms in the brackets of (4.32) are zeros. This gives us a system of linear equations for the diagonal elements such that

$$A_{ik,ik} = 0 \qquad 1 \leqslant i < k \leqslant n. \tag{4.38}$$

Hence, in this case the matrix A is nilpotent. If we have less or equal to $n-2$ off-diagonal elements, then we obtain at least one free element on the diagonal. In this case the matrix A can (and must) be chosen to be non-nilpotent.

(2) Using transformation (4.37) we normalize all $m \leqslant n-2$ nonzero off-diagonal elements to $+1$. This imposes a system of $n-2$ algebraic constraints on the $n-1$ coefficients $g_{i(i+1)}, i = 1, \ldots, n-1$. These equations always have a solution, but in some cases the solution may be complex. For $\mathbb{K} = \mathbb{C}$ this is consistent. For $\mathbb{K} = \mathbb{R}$ we must modify the initial normalized systems and include the possibility of normalizing to $+1$, or -1 the off-diagonal elements. Thus an equivalence class over $\mathbb{C}$ may be split into several over $\mathbb{R}$ (as usual, when restricting from the algebraicly closed field $\mathbb{C}$ to the nonclosed one $\mathbb{R}$). □

4.4. The Lie algebra $L(n, n-1)$

Let us now consider the other extreme case, namely $f = n-1$.

Lemma 6. The Lie algebra $L(n, n-1)$ has the following properties.

(1) Only one such Lie algebra exists. The structure matrices A^α are all diagonal and can be chosen to satisfy

$$A^\alpha_{ik,ab} = \delta_{ik,ab} \sum_{p=i}^{k-1} \delta_{\alpha p} \qquad 1 \leqslant i < k \leqslant n,\ 1 \leqslant \alpha \leqslant n-1. \tag{4.39}$$

(2) The non-nilpotent elements always commute, i.e.

$$[X^\alpha, X^\beta] = 0 \qquad 1 \leqslant \alpha, \beta \leqslant n-1. \tag{4.40}$$

Proof. Let us prove each statement in the theorem separately.

(1) From lemma 1, we see that by appropriate linear combinations of the elements X^α, we can choose A^α to satisfy

$$A^\alpha_{i(i+1),i(i+1)} = \delta_{\alpha i} \qquad \alpha, i = 1, \ldots, n-1. \tag{4.41}$$

By using point (3) of lemma 1, we obtain (4.39). To complete the proof we have to show that the off-diagonal elements are zero for all the structure matrices A^α. First, we use transformation (4.32) to annul the off-diagonal elements of A^1. Commutativity among the structure matrices then implies that the off-diagonal elements of $\{A^2, \ldots, A^{n-1}\}$ also vanish.

(2) The structure matrices given by (4.39) satisfy $A^\alpha_{1n,1n} = 1 \ \forall\alpha$. Hence, from equation (4.28) we obtain equation (4.40). □

4.5. The main results

The results of section 4 constitute the principal result of this article and can be summed up as a theorem.

Theorem 2. Every solvable Lie algebra $L(n, f)$ with a triangular nilradical $T(n)$ has the dimension $d = f + \frac{1}{2}n(n-1)$ with $1 \leqslant f \leqslant n-1$. It can be transformed to a canonical basis $\{X^\alpha, N_{ik}\}, \alpha = 1, \ldots, f,\ 1 \leqslant i < k \leqslant n$ with commutation relations

$$[N_{ik}, N_{ab}] = \delta_{ka}N_{ib} - \delta_{bi}N_{ak} \qquad [X^\alpha, N_{ik}] = A^\alpha_{ik,pq}N_{pq} \qquad [X^\alpha, X^\beta] = \sigma^{\alpha\beta}N_{1n}.$$

The canonical forms of the structure matrices A^α and the constants $\sigma^{\alpha\beta}$ satisfy the following conditions.

(1) The matrices A^α are linearly nilindependent and have the form specified in lemma 1. For $f \geqslant 2$ they all commute, i.e. $[A^\alpha, A^\beta] = 0$.

(2) All constants $\sigma^{\alpha\beta}$ vanish unless we have $A^\gamma_{1n,1n} = 0$ for $\gamma = 1, \ldots, f$.

(3) The remaining off-diagonal elements $A^\alpha_{ik,ab}$ also vanish, unless the diagonal elements satisfy $A^\beta_{ik,ik} = A^\beta_{ab,ab}$ for $\beta = 1, \ldots, f$.

(4) When f reaches its maximal value $f = n-1$, then all matrices A^α are diagonal as in equation (4.39) and all elements X^α commute.

(5) For $f = 1$ the matrix A^1 has at most $n-2$ off-diagonal elements that can be normalized as in lemma 5.

5. Conclusions

In this article we have provided a description and classification of solvable Lie algebras with triangular nilradicals $T(n)$. They are nilpotent Lie algebras that are in some sense, the furthest removed from Abelian algebras. Indeed, the dimensions of the Lie algebras in their central series are:

$$\dim CS : \left(\frac{n(n-1)}{2}, \frac{(n-1)(n-2)}{2}, \frac{(n-2)(n-3)}{2}, \ldots, 3, 1, 0\right).$$

This complements earlier work [9, 10] on the classification of solvable Lie algebras with Heisenberg nilradicals (with $\dim CS : (2n+1, 1, 0)$) and Abelian nilradicals (with $\dim CS : (n, 0)$).

The main results obtained in this paper are summed up in theorem 2 of section 4.

Applications of these algebras are postponed to a forthcoming article. They will concern Lie theory and differential equations. The algebras $L(n, f)$ can appear as symmetry algebras of nonlinear differential equations [11, 12]. They will also be used to construct certain nonlinear ordinary differential equations with superposition formulae [13–15].

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Appendix. Lie algebras with a six-dimensional triangular nilradical $T(4)$

As an illustration of the results obtained above, we give a list of all algebras $L(4, 1)$, $L(4, 2)$ and $L(4, 3)$.

We characterize the algebra by the structural matrices A^{α} and by the constants $\sigma^{\alpha\beta}$. For each algebra, we introduce a name $K_{f,i}(a, b)$ or $R_{f,i}(a, b)$. The letter K indicates that the algebra exists both for $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$; the algebra R is equivalent to some other algebra in the list for $\mathbb{K} = \mathbb{C}$, but inequivalent for $\mathbb{K} = \mathbb{R}$. The first subscript f indicates the number of nonnilpotent elements and the second subscript simply enumerates the algebras. The labels in the brackets indicate parameters in the matrices A^{α}. Note that for each matrix we normalize the first nonzero element on the diagonal to $+1$.

Table A.1. The Lie algebras $L(4, 1)$

Name	A	Parameters
$K_{1,1}(a, b)$	$\begin{pmatrix} 1 &&&&& \\ & a &&&& \\ && b &&& \\ &&& 1+a && \\ &&&& a+b & \\ &&&&& 1+a+b \end{pmatrix}$	$a, b \in \mathbb{K}$
$K_{1,2}(a)$	$\begin{pmatrix} 0 &&&&& \\ & 1 &&&& \\ && a &&& \\ &&& 1 && \\ &&&& 1+a & \\ &&&&& 1+a \end{pmatrix}$	$a \in \mathbb{K}$
$K_{1,3}$	$\begin{pmatrix} 0 &&&&& \\ & 0 &&&& \\ && 1 &&& \\ &&& 0 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}$	
$K_{1,4}(a)$	$\begin{pmatrix} 1 &&&& 1 & \\ & a &&&& \\ && 1-a &&& \\ &&& 1+a && \\ &&&& 1 & \\ &&&&& 2 \end{pmatrix}$	$a \in \mathbb{K}$

Table A.1. (Continued)

Name	A	Parameters
$K_{1,5}$	$\begin{pmatrix} 0 & & & & 1 & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$	
$K_{1,6}(a)$	$\begin{pmatrix} 1 & & & & & \\ & a & & & & 1 \\ & & -1 & & & \\ & & & 1+a & & \\ & & & & -1+a & \\ & & & & & a \end{pmatrix}$	$a \in \mathbb{K}$
$K_{1,7}$	$\begin{pmatrix} 0 & & & & & \\ & 1 & & & & 1 \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$	
$K_{1,8}(a)$	$\begin{pmatrix} 1 & & & & & \\ & a & & & & \\ & & 1+a & 1 & & \\ & & & 1+a & & \\ & & & & 1+2a & \\ & & & & & 2(1+a) \end{pmatrix}$	$a \in \mathbb{K}$
$K_{1,9}$	$\begin{pmatrix} 0 & & & & & \\ & 1 & & & & \\ & & 1 & 1 & & \\ & & & 1 & & \\ & & & & 2 & \\ & & & & & 2 \end{pmatrix}$	
$K_{1,10}$	$\begin{pmatrix} 1 & & & & & \\ & -2 & & & & 1 \\ & & -1 & 1 & & \\ & & & -1 & & \\ & & & & -3 & \\ & & & & & -2 \end{pmatrix}$	
$K_{1,11}$	$\begin{pmatrix} 1 & & & & 1 & \\ & 2 & & & & 1 \\ & & -1 & & & \\ & & & 3 & & \\ & & & & 1 & \\ & & & & & 2 \end{pmatrix}$	
$K_{1,12}$	$\begin{pmatrix} 1 & & & & 1 & \\ & 0 & & & & \\ & & 1 & 1 & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 2 \end{pmatrix}$	

Table A.1. (Continued)

Name	A	Parameters
$R_{1,13}$	$\begin{pmatrix} 1 & & & & 1 & \\ & 0 & & & & \\ & & 1 & -1 & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 2 \end{pmatrix}$	

Table A.2. The Lie algebras $L(4, 2)$.

Name	σ	A^1	A^2	Parameters
$K_{2,1}(a, b)$	0	$\begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & a & & & \\ & & & 1 & & \\ & & & & a & \\ & & & & & 1+a \end{pmatrix}$	$\begin{pmatrix} 0 & & & & & \\ & 1 & & & & \\ & & b & & & \\ & & & 1 & & \\ & & & & 1+b & \\ & & & & & 1+b \end{pmatrix}$	$a, b \in \mathbb{K}$
$K_{2,2}$	σ	$\begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$	$\sigma \in \mathbb{K} \setminus \{0\}$
$K_{2,3}(a)$	0	$\begin{pmatrix} 1 & & & & & \\ & a & & & & \\ & & 0 & & & \\ & & & 1+a & & \\ & & & & a & \\ & & & & & 1+a \end{pmatrix}$	$\begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$	$a \in \mathbb{K}$
$K_{2,4}$	0	$\begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$	
$K_{2,5}(a)$	0	$\begin{pmatrix} 1 & & & & & \\ & a & & & & \\ & & 1-a & & & \\ & & & 1+a & & \\ & & & & 1 & \\ & & & & & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & & & & 1 & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$	$a \in \mathbb{K}$
$K_{2,6}$	0	$\begin{pmatrix} 0 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & & & & 1 & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 2 \end{pmatrix}$	

Table A.2. (Continued)

Name	σ	A^1	A^2	Parameters
$K_{2,7}(a)$	0	$\begin{pmatrix} 1 &&&&& \\ & a &&&& \\ && -1 &&& \\ &&& 1+a && \\ &&&& -1+a & \\ &&&&& a \end{pmatrix}$	$\begin{pmatrix} 0 &&&&& \\ & 1 &&&& 1 \\ && 0 &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}$	$a \in \mathbb{K}$
$K_{2,8}$	0	$\begin{pmatrix} 0 &&&&& \\ & 1 &&&& \\ && 0 &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}$	$\begin{pmatrix} 1 &&&&& \\ & 0 &&&& 1 \\ && -1 &&& \\ &&& 1 && \\ &&&& -1 & \\ &&&&& 0 \end{pmatrix}$	
$K_{2,9}(a)$	0	$\begin{pmatrix} 1 &&&&& \\ & a &&&& \\ && 1+a &&& \\ &&& 1+a && \\ &&&& 1+2a & \\ &&&&& 2(1+a) \end{pmatrix}$	$\begin{pmatrix} 0 &&&&& \\ & 1 &&&& \\ && 1 & 1 && \\ &&& 1 && \\ &&&& 2 & \\ &&&&& 2 \end{pmatrix}$	$a \in \mathbb{K}$
$K_{2,10}$	0	$\begin{pmatrix} 0 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& 2 & \\ &&&&& 2 \end{pmatrix}$	$\begin{pmatrix} 1 &&&&& \\ & 0 &&&& \\ && 1 & 1 && \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 2 \end{pmatrix}$	

Table A.3. The Lie algebra $L(4, 3)$.

Name	$\sigma' s$	A^1	A^2	A^3
$K_{3,1}$	0	$\begin{pmatrix} 1 &&&&& \\ & 0 &&&& \\ && 0 &&& \\ &&& 1 && \\ &&&& 0 & \\ &&&&& 1 \end{pmatrix}$	$\begin{pmatrix} 0 &&&&& \\ & 1 &&&& \\ && 0 &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}$	$\begin{pmatrix} 0 &&&&& \\ & 0 &&&& \\ && 1 &&& \\ &&& 0 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}$

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